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On the flow of a rotating mixture in a sectioned cylinder

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We consider the centrifugal separation of an initially homogeneous mixture in an asymmetric geometry. The mixture is shown to acquire a uniform and negative relative vorticity, manifested as a retrograde circulation, as the heavier phase is forced outwards by the centrifugal acceleration. The perturbation theory formulated accounts for inertial effects and endwall boundary layers as well as slight deviations of endwall shape from a level plane. The time-dependent separation process is described and the flow field, including kinematic shocks, is calculated in some cases of technological interest.

1. Introduction

We consider centrifugal batch separation of a two-phase mixture which consists of a dispersed phase of fluid droplets or solid particles within a continuous Newtonian liquid. This problem has been the subject of several recent studies, all of which use variations of the same basic mathematical model and place emphasis on the effects of geometrical modifications of the container and of different parameter regimes.

Greenspan (1983) considered an infinite circular cylinder and presented an exact similarity solution to the full nonlinear two-phase flow equations. It was found that the mixture, which initially rotated with the cylinder, is set in a retrograde rotation relative to the cylinder as separation occurs. This is caused by the outward radial mass flux during separation which is accompanied by a decreasing angular velocity in the mixture in order to conserve angular momentum. Except for the drag between the phases, viscous phenomena, such as endwall effects, were neglected in the analysis.

The effects of the Ekman layers at the endwalls were studied by Ungarish (1986) under the assumption that the Rossby number and the particle Taylor number are small. The retrograde rotation of the mixture is then counteracted by the spin-up mechanism due to the secondary flow set up by the Ekman-layer suction. If the spin-up timescale is much shorter than that for separation, the retrograde rotation of the mixture is strongly damped but the interior radial velocities of the phases are quite insensitive to the influence of the boundary layers.

To investigate the possibility of an enhanced settling rate similar to the Boycott effect in gravitational settling, Greenspan & Ungarish (1985*a*, *b*) considered an axisymmetric container with inclined endwalls. It was shown that an enhanced mass flux, which ordinarily appears owing to buoyancy forces in the clear-fluid layer adjacent to an inclined wall, is in a rotating container blocked by the Coriolis force which dominates the interior force balance. Another effect of the Coriolis force in an axisymmetric container is that the retrograde rotation of the mixture increases with increasing values of the relative density difference between the phases and thereby reduces the effective centrifugal force on the heavier phase. To decrease the negative effects of the Coriolis force, a disk stack with small gap thickness is used in industrial design (Amberg *et al.* 1986); alternatively, motion around the axis of rotation can be prevented by inserting a radial wall in the cylinder. Greenspan & Ungarish (1985*a*) calculated the enhanced settling rate due to such a barrier based on simplifying assumptions about the viscous boundary layers at inclined endwalls. A more careful investigation of the structure and transport mechanisms in the boundary layers was made by Amberg & Greenspan (1986).

An enhanced settling rate can be achieved even without inclined endwalls. At relatively large particle Taylor numbers, the Coriolis force on the particles due to the relative motion between the phases is significant, and according to Greenspan (1983) a sedimenting particle then has a substantial component of velocity in the azimuthal direction. As demonstrated by Schaflinger, Köppl & Filipzak (1986), who considered an infinitely long cylinder with radial walls, the result in this case is an enhanced settling rate caused by the production of clear fluid adjacent to each sectorial barrier. They showed, theoretically and experimentally, that for moderately large values of the particle Taylor number the clear fluid produced in this way can significantly decrease the time for total separation. (This theoretical analysis, however, assumes the Rossby number to be zero and the force balance to be purely hydrostatic.)

The aim here is a better understanding of the total flow in a separating mixture within a finite, non-axisymmetric cylinder. The theoretical formulation and analytical development presented can account in a systematic way for the effects of endwall boundary layers and geometry, inertial and geostrophic dynamics, vorticity production from density stratification, and arbitrary initial conditions.

2. Formulation

Separation is assumed to take place in a rotating sectioned cylinder of height H^*, inner radius r_i^*, outer radius r_o^* and sector angle Θ. A cylindrical coordinate system is used that rotates with the container at angular velocity Ω^* around the z-axis (see figure 1). The mixture consists of a continuous liquid phase and a homogeneous dispersed phase of spherical particles all with the same radius a^*. The 'mixture' (or 'diffusion') model for two-phase flow is used (see Ishii 1975) where the mass-averaged velocity of the mixture is denoted by $\boldsymbol{q}^* = (q_r^*, q_\theta^*, q_z^*)$ and the corresponding volume flux by $\boldsymbol{j}^* = (j_r^*, j_\theta^*, j_z^*)$. Variables of the continuous and the dispersed phases are denoted by subscripts C, D as for example, densities ρ_C^* and ρ_D^* and the viscosity of the liquid ν_C^*. In terms of the particle volume fraction α, the mixture density is

$$\rho^* = \alpha\rho_D^* + (1-\alpha)\rho_C^*; \tag{2.1}$$

the velocity difference between the phases is denoted by

$$\boldsymbol{q}_R^* = \boldsymbol{q}_D^* - \boldsymbol{q}_C^*. \tag{2.2}$$

Non-dimensional variables are defined as follows:

$$\boldsymbol{r} = \frac{\boldsymbol{r}^*}{r_o^*}, \quad t = t^*|\epsilon|\beta\Omega^*, \quad \boldsymbol{q}_R = \frac{\boldsymbol{q}_R^*}{|\epsilon|\beta\Omega^* r_o^*}, \quad \boldsymbol{q} = \frac{\boldsymbol{q}^*}{Ro\,\Omega^* r_o^*}, \quad P = \frac{P^*/\rho_C^*(\Omega^* r_o^*)^2 - \frac{1}{2}r^2}{Ro},$$

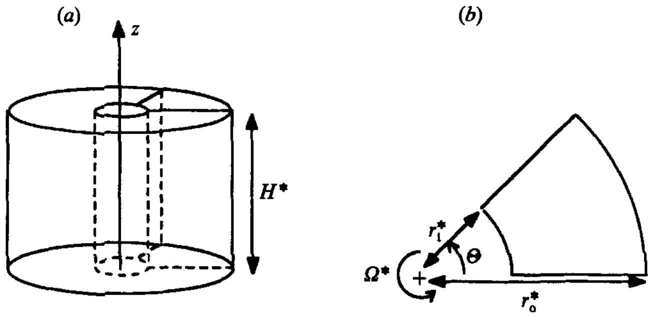


FIGURE 1. Geometry of the container: (*a*) side view; (*b*) top view.

where P^* is the averaged mixture pressure. The relative density difference ϵ and particle Taylor number β are dimensionless parameters defined by

$$\epsilon = \frac{\rho_D^* - \rho_C^*}{\rho_C^*}, \quad \beta = \frac{2a^{*2}}{9\nu_C^*/\Omega^*}.$$

The Ekman number is

$$E = \frac{\nu_C^*}{\Omega^* r_o^{*2}},$$

and a suitable Rossby number *Ro* will be specified later.

For reference, some relations between velocities, volume fluxes and the relative velocity are

$$\boldsymbol{q}_D = \boldsymbol{q} + \frac{|\epsilon|\beta(1-\alpha)}{Ro(1+\epsilon\alpha)}\boldsymbol{q}_R, \tag{2.3a}$$

$$\boldsymbol{j}_D = \alpha\boldsymbol{j} + \frac{|\epsilon|\beta}{Ro}\alpha(1-\alpha)\boldsymbol{q}_R, \tag{2.3b}$$

$$\boldsymbol{j} = \boldsymbol{q} - \epsilon\frac{|\epsilon|\beta\alpha(1-\alpha)}{Ro(1+\epsilon\alpha)}\boldsymbol{q}_R. \tag{2.3c}$$

The mixture theory for two-phase flow consists of two mass conservation equations for the incompressible constituents:

$$\nabla\cdot\boldsymbol{j} = 0, \tag{2.4}$$

$$|\epsilon|\beta\frac{\partial\alpha}{\partial t} + Ro\,\nabla\cdot\boldsymbol{j}_D = 0, \tag{2.5}$$

and a single momentum equation for the velocity field of the bulk

$$(1+\epsilon\alpha)\left(|\epsilon|\beta\frac{\partial\boldsymbol{q}}{\partial t} + Ro\,\boldsymbol{q}\cdot\nabla\boldsymbol{q}\right) + (1+\epsilon\alpha)\,2\boldsymbol{k}\times\boldsymbol{q} = -\nabla P - \frac{\epsilon\alpha}{Ro}\boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{r})$$
$$-\frac{|\epsilon|^2\beta^2}{Ro}(1+\epsilon)\nabla\cdot\frac{\alpha(1-\alpha)}{1+\epsilon\alpha}\boldsymbol{q}_R\boldsymbol{q}_R + D(\alpha)\,E\{\tfrac{4}{3}\nabla(\nabla\cdot\boldsymbol{q}) - \nabla\times\nabla\times\boldsymbol{q}\}. \tag{2.6}$$

These are supplemented by a constitutive law for the relative velocity

$$sA(\alpha)\left[|\epsilon|\beta\,Ro\frac{\partial\boldsymbol{q}}{\partial t} + Ro^2\boldsymbol{q}\cdot\nabla\boldsymbol{q} + Ro\,2\boldsymbol{k}\times\boldsymbol{q} + \boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{r})\right] = -\boldsymbol{q}_R - B(\alpha)\,\boldsymbol{k}\times\boldsymbol{q}_R \tag{2.7}$$

where
$$A(\alpha) = \frac{1-\alpha}{D(\alpha)}, \quad B(\alpha) = \frac{2\beta(1+\epsilon)A}{1+\epsilon\alpha}, \quad s = \frac{\epsilon}{|\epsilon|}, \tag{2.8}$$

and an empirical formula for the effective viscosity, as for example

$$D(\alpha) = (1-\alpha/\alpha_M)^{-2.5\alpha_M} \tag{2.9}$$

(with $\alpha_M = 0.66$ for computational purposes). The right-hand side of (2.7) consists of the Stokesian drag and the Coriolis force on a particle due to the relative velocity between the phases; the left-hand side models the centrifugal force on a particle as modified by inertial effects due to the motion of the mixture. In (2.6) and (2.7) gravity is neglected owing to the rapid rotation of the cylinder.

For the most part the mixture is assumed to be initially homogeneous and in rigid rotation:

$$\alpha(t=0) = \alpha_i, \quad \boldsymbol{q}(t=0) = 0. \tag{2.10}$$

No initial condition can be prescribed on the relative velocity $\boldsymbol{q}_R$ which is assumed to be quasi-steady. If a time derivative of $\boldsymbol{q}_R$ is included in the constitutive law (2.7) (see Greenspan 1988) the initial-value problem for $\boldsymbol{q}_R$ is well posed, in which case the relative velocity attains its quasi-steady value on a timescale much shorter than that used here.

With conditions of no slip imposed on the walls, (2.4), (2.5) and (2.6) together with (2.7) and (2.10) constitute a well set initial- and boundary-value problem for the dependent variables $\boldsymbol{q}$ and α. This problem will be studied under the assumptions that the Ekman number E and the relative density difference ϵ are small but non-zero. The particle Taylor number β, which can be interpreted as the square of the ratio of the particle radius and the Ekman-layer thickness, is often small too in applications but is here treated as an order-one quantity in a formal perturbation procedure. However, it is clear that if β is too large the mixture model breaks down in the Ekman layers. Also, it should be pointed out that for large β the constitutive law (2.7) must be modified owing to the change in character of the local flow field around a particle. Since E and ϵ are small, the interior flow of the mixture is mainly inviscid and linear boundary-layer theory can be used to obtain boundary conditions for the interior flow (Greenspan 1968). It follows that at the horizontal top and bottom plates

$$q_z(z=0) = \tfrac{1}{2}E^{\frac{1}{2}}[D(\alpha)]^{\frac{1}{2}}\left(\omega + \nabla\cdot\boldsymbol{q} - \frac{\partial q_z}{\partial z}\right)\bigg|_{z=0}, \tag{2.11a}$$

$$q_z(z=H) = -\tfrac{1}{2}E^{\frac{1}{2}}[D(\alpha)]^{\frac{1}{2}}\left(\omega + \nabla\cdot\boldsymbol{q} - \frac{\partial q_z}{\partial z}\right)\bigg|_{z=H}, \tag{2.11b}$$

where $\omega = \boldsymbol{k}\cdot(\nabla\times\boldsymbol{q})$, while at the vertical walls

$$\boldsymbol{j}\cdot\boldsymbol{n} = O(E^{\frac{1}{2}}). \tag{2.12}$$

These formulas also assume the volume fraction to be constant in the Ekman layers (Ungarish 1986).

The problem as formulated generally leads to solutions that involve kinematic shocks to separate regions of clarified fluid, suspension and sediment. If the radial position of a shock is assumed to be given by $r = R(\theta, t)$, then the shock conditions for the conservation of momentum, mass and particle volume and the no-slip conditions for a viscous fluid are respectively

$$-\left[(1+\epsilon\alpha)\,\boldsymbol{q}\left(|\epsilon|\,\beta\frac{\partial R}{\partial t}-|\epsilon|\,\boldsymbol{q}\cdot\boldsymbol{n}_R\right)\right]_-^+ = -[P]_-^+\boldsymbol{n}_R+E[\boldsymbol{T}\cdot\boldsymbol{n}_R]_-^+ - \left[|\epsilon|\,\beta^2(1+\epsilon)\frac{\alpha(1-\alpha)}{1+\epsilon\alpha}\,\boldsymbol{q}_{\mathrm{R}}\,\boldsymbol{q}_{\mathrm{R}}\cdot\boldsymbol{n}_R\right]_-^+; \quad (2.13a)$$

$$\left[(1+\epsilon\alpha)\left(|\epsilon|\,\beta\frac{\partial R}{\partial t}-|\epsilon|\,\boldsymbol{q}\cdot\boldsymbol{n}_R\right)\right]_-^+ = 0; \quad (2.13b)$$

$$\left[\alpha\left(|\epsilon|\,\beta\frac{\partial R}{\partial t}-\left(|\epsilon|\,\boldsymbol{q}+|\epsilon|\,\beta\frac{1-\alpha}{1+\epsilon\alpha}\,\boldsymbol{q}_{\mathrm{R}}\right)\cdot\boldsymbol{n}_R\right)\right]_-^+ = 0; \quad (2.13c)$$

$$[\boldsymbol{q}\cdot\boldsymbol{k}]_-^+ = 0; \quad (2.13d)$$

$$[\boldsymbol{q}\cdot(\boldsymbol{k}\times\boldsymbol{n}_R)]_-^+ = 0; \quad (2.13e)$$

where

$$\boldsymbol{n}_R = \boldsymbol{e}_r-\frac{1}{R}\frac{\partial R}{\partial\theta}\,\boldsymbol{e}_\theta, \quad (2.13f)$$

and $\boldsymbol{T}$ is the viscous stress tensor. Here we will deal only with the interface separating the clear fluid from the suspension and all effects of accumulated sediment on the container walls will be neglected in the analysis. Therefore the analysis is valid only for small concentrations when the sediment layer is thin. With $\epsilon > 0$, $\alpha^- = 0$ and $\alpha^+ = \alpha$, it follows that

$$-[P]_-^+\boldsymbol{n}_R\cdot\boldsymbol{n}_R+E[\boldsymbol{n}_R\cdot\boldsymbol{T}\cdot\boldsymbol{n}_R]_-^+ = \epsilon\beta^2\alpha(1-\alpha)\,(\boldsymbol{q}_{\mathrm{R}}^+\cdot\boldsymbol{n}_R)^2; \quad (2.14a)$$

$$[\boldsymbol{q}\cdot\boldsymbol{n}_R]_-^+ = \epsilon\beta\frac{\alpha(1-\alpha)}{1+\epsilon\alpha}\,\boldsymbol{q}_{\mathrm{R}}^+\cdot\boldsymbol{n}_R; \quad (2.14b)$$

$$\beta\frac{\partial R}{\partial t}-\boldsymbol{q}^+\cdot\boldsymbol{n}_R = \beta\frac{1-\alpha}{1+\epsilon\alpha}\,\boldsymbol{q}_{\mathrm{R}}^+\cdot\boldsymbol{n}_R; \quad (2.14c)$$

$$-\epsilon\beta(1-\alpha)\,\boldsymbol{q}_{\mathrm{R}}^+\cdot\boldsymbol{n}_R[\boldsymbol{q}\cdot\boldsymbol{k}]_-^+ = E[\boldsymbol{k}\cdot\boldsymbol{T}\cdot\boldsymbol{n}_R]_-^+; \quad (2.14d)$$

$$-\epsilon\beta(1-\alpha)\,\boldsymbol{q}_{\mathrm{R}}^+\cdot\boldsymbol{n}_R[\boldsymbol{q}\cdot(\boldsymbol{k}\times\boldsymbol{n}_R)]_-^+ = E[(\boldsymbol{k}\times\boldsymbol{n}_R)\cdot\boldsymbol{T}\cdot\boldsymbol{n}_R]_-^+ - \epsilon\beta^2(1+\epsilon)\frac{\alpha(1-\alpha)}{1+\epsilon\alpha}\,\boldsymbol{q}_{\mathrm{R}}^+\cdot(\boldsymbol{k}\times\boldsymbol{n}_R)\,\boldsymbol{q}_{\mathrm{R}}^+\cdot\boldsymbol{n}_R; \quad (2.14e)$$

$$[\boldsymbol{q}\cdot\boldsymbol{k}]_-^+ = 0; \quad [\boldsymbol{q}\cdot(\boldsymbol{k}\times\boldsymbol{n}_R)]_-^+ = 0. \quad (2.14f,g)$$

For an inviscid fluid the no-slip conditions are omitted and with $E = 0$ the second term in (2.14a) is neglected and (2.14d–g) reduce to

$$[\boldsymbol{q}\cdot\boldsymbol{k}]_-^+ = 0; \quad (2.15a)$$

$$[\boldsymbol{q}\cdot(\boldsymbol{k}\times\boldsymbol{n}_R)]_-^+ = \beta\frac{(1+\epsilon)\,\alpha}{1+\epsilon\alpha}\,\boldsymbol{q}_{\mathrm{R}}^+\cdot(\boldsymbol{k}\times\boldsymbol{n}_R). \quad (2.15b)$$

Using the definition (2.2) and the relation (2.3a), it follows from (2.15a) and (2.15b) that the tangential components of the clear-fluid velocity are continuous across the interface. Thus, (2.15a, b) are consequences of the inertia of the clear fluid that passes from the suspension into the clear-fluid region across the interface.

3. The vorticity equation

The vorticity of the mass-averaged velocity field is used in the analysis of the process. The curl of the inviscid form of (2.6) yields

$$(1+\epsilon\alpha)\left[|\epsilon|\beta\frac{\partial\omega}{\partial t}+Ro(\boldsymbol{q}\cdot\nabla\omega-(\nabla\times\boldsymbol{q})\cdot\nabla q_z)\right]$$
$$+\epsilon\boldsymbol{k}\times\nabla\alpha\cdot\left[|\epsilon|\beta\frac{\partial\boldsymbol{q}}{\partial t}+Ro\,\boldsymbol{q}\cdot\nabla\boldsymbol{q}\right]+2\epsilon\nabla\alpha\times(\boldsymbol{k}\times\boldsymbol{q})+2(1+\epsilon\alpha)\nabla\cdot\boldsymbol{q}-2(1+\epsilon\alpha)\frac{\partial q_z}{\partial z}$$
$$=-\frac{\epsilon}{Ro}\frac{\partial\alpha}{\partial\theta}-\frac{|\epsilon|^2\beta^2}{Ro}(1+\epsilon)\boldsymbol{k}\cdot\left(\nabla\times\left[\nabla\cdot\frac{\alpha(1-\alpha)}{1+\epsilon\alpha}\boldsymbol{q}_{\mathrm{R}}\boldsymbol{q}_{\mathrm{R}}\right]\right), \quad (3.1)$$

where
$$\omega=\boldsymbol{k}\cdot(\nabla\times\boldsymbol{q}).$$

By examining this equation an appropriate Rossby number can be chosen to scale the velocity field. It is obvious that the vorticity generated depends in part on the stratification of the mixture, that is on baroclinic processes. If there is no stratification in the vertical direction, the z-component of the velocity can be expected to be small, of magnitude given by the Ekman-layer suction, i.e. proportional to $E^{\frac{1}{2}}$. Large buoyant forces due to stratification in the horizontal plane must then be balanced by inertia rather than by vortex stretching, which, by balancing the second term on the left-hand side with the first term on the right-hand side, implies $Ro=O(|\epsilon|^{\frac{1}{2}})$. However, even if the mixture happens to be homogeneous in space, vorticity is still generated because of the divergent velocity field due to the sedimenting particles. Since from (2.4) and (2.3), we have that

$$\nabla\cdot\boldsymbol{q}=\epsilon\frac{|\epsilon|\beta}{Ro}\nabla\cdot\frac{\alpha(1-\alpha)}{(1+\epsilon\alpha)}\boldsymbol{q}_{\mathrm{R}}, \quad (3.2)$$

the balance of the divergence by the inertia term in (3.1) implies $Ro=O(|\epsilon|)$. This Rossby number will also apply to the situation in which the stratification is non-zero in the radial direction but at most of order ϵ in the azimuthal direction.

The effect of the strong coupling between vorticity and stratification in the mixture on the volume fraction of particles requires discussion. Equation (2.5) can be written

$$\frac{\partial\alpha}{\partial t}+\left\{\frac{Ro}{|\epsilon|\beta}\boldsymbol{q}+(1+\epsilon\alpha)\frac{\partial}{\partial\alpha}\left(\frac{\alpha(1-\alpha)}{1+\epsilon\alpha}\boldsymbol{q}_{\mathrm{R}}\right)\right\}\cdot\nabla\alpha=-\alpha(1-\alpha)\nabla\cdot\boldsymbol{q}_{\mathrm{R}}|_\alpha \quad (3.3)$$

where $|_\alpha$ means that α should be held constant when differentiating. The relative velocity is obtained from the constitutive law (2.7)

$$\boldsymbol{q}_{\mathrm{R}}=s\frac{A(\alpha)}{1+B(\alpha)^2}[r(\boldsymbol{e}_r+B(\alpha)\boldsymbol{e}_r\times\boldsymbol{k})-2Ro\,\boldsymbol{k}\times\boldsymbol{q}-2Ro\,B(\alpha)(\boldsymbol{k}\times\boldsymbol{q})\times\boldsymbol{k}], \quad (3.4)$$

where second-order terms in Ro and ϵ have been neglected. To lowest order in Ro the components of the relative velocity are thus explicitly independent of the azimuthal coordinate. With α held constant the divergence of (3.4) gives

$$\nabla\cdot\boldsymbol{q}_{\mathrm{R}}|_\alpha=s\frac{2A(\alpha)}{1+B(\alpha)^2}(1+Ro\,\omega), \quad (3.5)$$

to this order of accuracy. For future use, note that

$$\nabla \times \boldsymbol{q}_{\mathrm{R}}|_{\alpha} = -\boldsymbol{k}B(\alpha)\,\nabla\cdot\boldsymbol{q}_{\mathrm{R}}|_{\alpha}. \tag{3.6}$$

Now, according to (3.3) concentration waves are convected by the mixture velocity, which in turn is determined by the generated vorticity field. For example, a stratification that is initially in the radial direction only cannot in general remain so in a non-axisymmetric container because the wave velocity components of (3.3) depend on the azimuthal coordinate even if the relative velocity components do not. In other words, a line of equal concentration will not propagate with the same radial velocity for all values of the azimuthal angle and therefore cannot remain circular. If there is not any initial stratification, as assumed in the formulation of the problem in the previous section, the convection of concentration waves will be of no importance if, on all characteristic paths, the concentration is the same. In an axisymmetric container it has been shown by Greenspan (1983) that this is actually the case even for finite Rossby numbers and the concentration then remains homogeneous and a function of time alone. Here, in the case of a non-axisymmetric cylinder, the volume fraction will be shown to be independent of space coordinates to first order in the Rossby number. The generated vorticity is then independent of the space coordinates too but only to lowest order in the Rossby number. Of course kinematic shocks may appear in the fluid to separate regions of clear fluid, suspension and sediment. These shocks have no influence on the volume fraction of particles in the suspension if the concentration is so small that no expansion waves propagate out from the sediment layer (Kynch 1952).

In the problem formulated in the previous section the concentration will remain nearly uniform. A suitable Rossby number is then

$$Ro = |\epsilon|, \tag{3.7}$$

or $|\epsilon|\,\alpha_{\mathrm{i}}$ if α_{i} is also small.

4. Analysis

Let

$$|\epsilon| = cE^{\frac{1}{2}} = Ro, \tag{4.1}$$

where c is an order-one quantity. It is natural to seek a solution in the form of a power series expansion in the small parameter $E^{\frac{1}{2}}$ in which case any dependent variable $y(\boldsymbol{r},t)$ is represented as

$$y(\boldsymbol{r},t) = y^0(\boldsymbol{r},t) + E^{\frac{1}{2}}y^1(\boldsymbol{r},t) + O(E), \tag{4.2}$$

where the quantities $y^{0,1}$ are assumed to be of order unity. The substitution of the power series expansions into the governing equations yields to zeroth order:

$$\boldsymbol{j}^0 = \boldsymbol{q}^0; \tag{4.3}$$

$$\nabla\cdot\boldsymbol{q}^0 = 0, \tag{4.4}$$

$$\frac{\mathrm{d}\alpha^0}{\mathrm{d}t} = -s2\alpha^0(1-\alpha^0)\frac{A^0}{1+(B^0)^2}, \tag{4.5}$$

$$2\boldsymbol{k}\times\boldsymbol{q}^0 = -\nabla P^0 + \alpha^0 r\boldsymbol{e}_r; \tag{4.6}$$

the vorticity equation gives

$$\frac{\partial q_z^0}{\partial z} = 0. \tag{4.7}$$

A^0 and B^0 are the zeroth-order expansions of the functions defined in (2.8):

$$A^0 = A(\alpha^0), \quad B^0 = 2\beta A(\alpha^0). \tag{4.8}$$

The boundary and initial conditions imply

$$\boldsymbol{q}^0 \cdot \boldsymbol{n} = 0 \tag{4.9}$$

at the vertical walls,

$$q_z^0 = 0 \tag{4.10}$$

at the top and bottom plates, and

$$\alpha^0(t=0) = \alpha_i, \quad \boldsymbol{q}^0(t=0) = 0. \tag{4.11}$$

The concentration α^0, a function of time alone, is easily obtained by integrating (4.5). The velocity field to zeroth order is then geostrophic and can be written as

$$\boldsymbol{q}^0 = \tfrac{1}{2}\boldsymbol{k} \times \boldsymbol{\nabla}\phi^0, \tag{4.12}$$

where

$$\phi^0 = P^0 - \alpha^0(t)\, r^2/2, \tag{4.13}$$

which becomes a primary dependent variable.

The geostrophic velocity is specified at the next order in the expansion of the vorticity equation:

$$\frac{2}{c\beta}\frac{\partial q_z^1}{\partial z} = \frac{\partial \omega^0}{\partial t} + \frac{1}{\beta}\boldsymbol{q}^0 \cdot \boldsymbol{\nabla}\omega^0 + s2\alpha^0(1-\alpha^0)\,\boldsymbol{\nabla}\cdot\boldsymbol{q}_{\mathrm{R}}^0 + 2\beta\alpha^0(1-\alpha^0)\,\boldsymbol{\nabla}\cdot\boldsymbol{q}_{\mathrm{R}}^0(\boldsymbol{k}\cdot\boldsymbol{\nabla}\times\boldsymbol{q}_{\mathrm{R}}^0). \tag{4.14}$$

The right-hand side is independent of the vertical coordinate and the integration of this equation, subject to the equivalent endwall conditions

$$q_z^1(z=0) = [D(\alpha^0)]^{\frac{1}{2}}\omega^0/2, \quad q_z^1(z=H) = -[D(\alpha^0)]^{\frac{1}{2}}\omega^0/2, \tag{4.15}$$

implies

$$\frac{\mathrm{D}\omega^0}{\mathrm{D}t} + 2[D(\alpha^0)]^{\frac{1}{2}}\lambda\omega^0 = -4\alpha^0(1-\alpha^0)\frac{A^0}{1+(B^0)^2} + 4\alpha^0(1-\alpha^0)A^0\frac{(B^0)^2}{[1+(B^0)^2]^2}, \tag{4.16}$$

where,

$$\frac{\mathrm{D}\omega^0}{\mathrm{D}t} = \frac{\partial\omega^0}{\partial t} + \frac{1}{\beta}\boldsymbol{q}^0\cdot\boldsymbol{\nabla}\omega^0. \tag{4.17}$$

Here,

$$\lambda = \frac{E^{\frac{1}{2}}}{|\epsilon|\,\beta H} = \frac{1}{c\beta H}, \tag{4.18}$$

is the ratio of the separation to the spin-up timescales. The right-hand side of (4.16) has no spatial dependence, and since

$$\omega^0(t=0) = 0, \tag{4.19}$$

the vorticity to zeroth order is a function only of time. (This conclusion is more generally valid, Greenspan 1988.) Numerical solutions of (4.16), using (4.5), are shown in figure 2. In figure 2(*a*) it is seen how the generated vorticity is damped as an effect of the spin-up mechanism for non-zero λ. For very large values of λ the process is quasi-steady. By neglecting the time derivative in (4.16), it follows that for large λ

$$\omega^0 = -\frac{1}{\lambda}\,2\alpha^0(1-\alpha^0)\frac{A^0}{1+(B^0)^2}[D(\alpha^0)]^{-\frac{1}{2}}\left(1 - \frac{(B^0)^2}{1+(B^0)^2}\right). \tag{4.20}$$

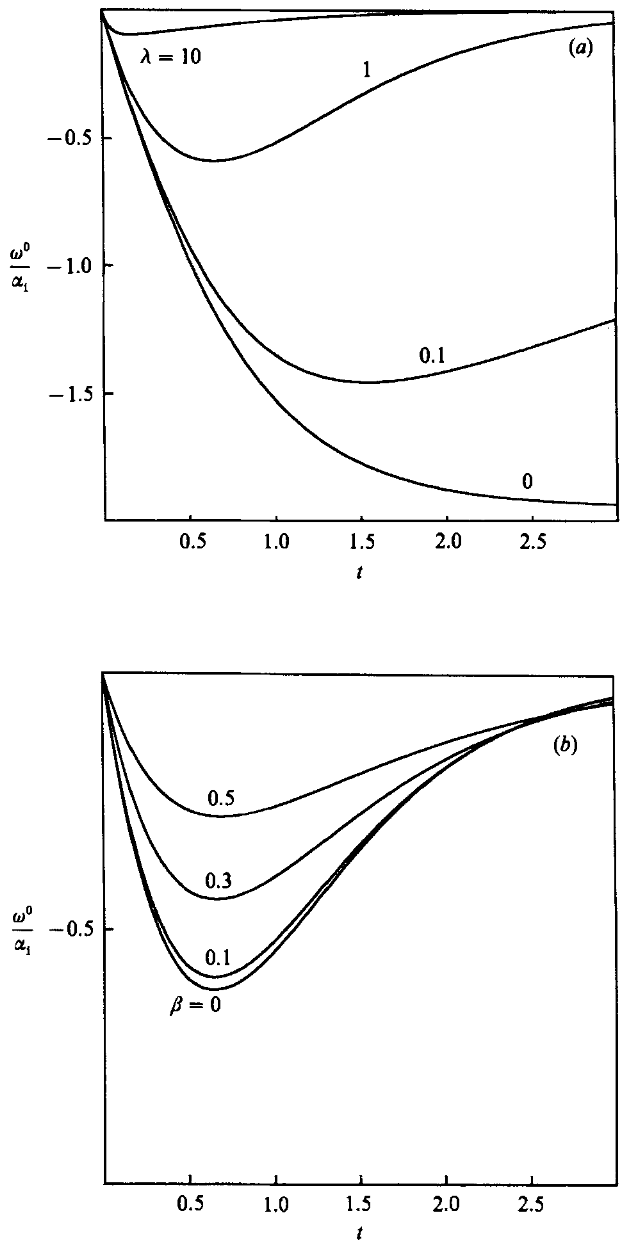


FIGURE 2. (*a*) Vorticity versus time for different values of λ: $\alpha_l = 0.1$, $\beta = 0.1$. (*b*) Vorticity versus time for different values of β: $\alpha_l = 0.1$, $\lambda = 1.0$.

Figure 2(*b*) shows that increasing the particle Taylor number β decreases the magnitude of the generated vorticity. This is caused by the Coriolis force on the particles which results in two different physical effects: the magnitude of the relative velocity decreases with increasing values of β in comparison with a purely centrifugal force field; there is a radial dispersal of momentum due to the diffusion stress tensor acting on the sedimenting particles. The latter effect is modelled by the last term in the right-hand side of (3.1).

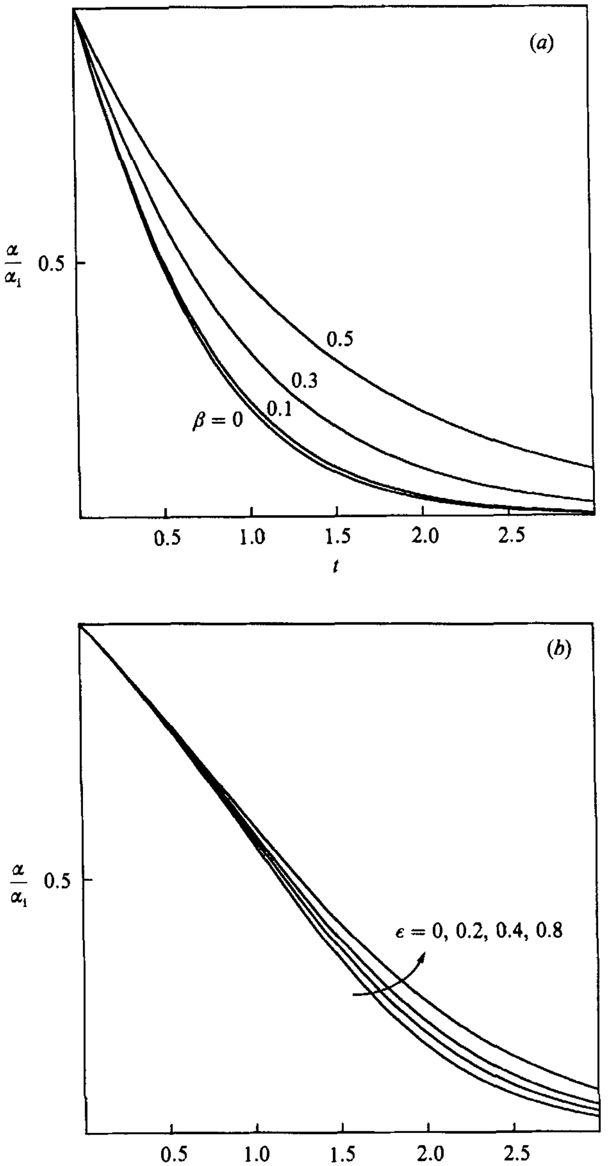


FIGURE 3. (a) α/α_i versus time for different values of β: $\alpha_i = 0.1$, $\lambda = 0.1$, $\epsilon = 0.2$. (b) α/α_i versus time for different values of $|\epsilon|$: $\alpha_i = 0.3$, $\lambda = 0.1$, $\beta = 0.1$.

With ω^0 given by (4.16), a first-order correction to the concentration can be obtained from (3.3):

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} = -s2\alpha(1-\alpha)\frac{A(\alpha)}{1+B(\alpha)^2}(1+|\epsilon|\,\omega^0(t)). \tag{4.21}$$

Since ω^0 is independent of the space coordinates, the concentration is a function of time also to first order. Rather than inserting the power series expansion for α, (4.21) is more easily solved as it stands to obtain

$$\alpha = \alpha^0(t) + E^{\frac{1}{2}}\alpha^1(t) + O(E). \tag{4.22}$$

Solutions to (4.21) are shown in figure 3. The effective Rossby number for small α is $\alpha_1|\epsilon|$ rather than $|\epsilon|$ so the result of increasing $|\epsilon|$ is rather moderate for dilute suspensions (see figure 3*b*).†

The first-order correction to the vorticity leads to an equation that involves $\alpha^2(\boldsymbol{r},t)$. This term, though, generally depends on the space coordinates because of nonlinear inertial effects in the equation for the relative velocity. Therefore the perturbation series imply a solution of the form

$$\left.\begin{aligned} \alpha &= \alpha^0(t)+\alpha^1(t)\,E^{\frac{1}{2}}+\alpha^2(\boldsymbol{r},t)\,E+O(E^{\frac{3}{2}}),\\ \omega &= \omega^0(t)+\omega^1(\boldsymbol{r},t)\,E^{\frac{1}{2}}+O(E). \end{aligned}\right\} \tag{4.23}$$

5. The flow field

According to (4.12), ϕ^0 is a stream function for the zeroth-order flow field, and the curl of this equation yields

$$\nabla^2\phi^0 = 2\omega^0, \tag{5.1}$$

where ω^0 is given by the solution of (4.16). Since ω^0, a function of time only, can be obtained irrespective of the velocity field, ϕ^0 can be calculated from (5.1) as soon as the boundary conditions and the geometry of the suspension region are specified. However, the vorticity is generally spatially dependent in the region of clarified fluid and the convection of vorticity then requires solution of the coupled system of equations (5.1) (4.16).

Consider first the simple case of a container that has no inner cylindrical boundary, figure 1, so that no clear fluid is produced by heavy particles moving away from an interior surface. The angular component of the relative velocity indicates that clear fluid may indeed appear at the radial walls, but this is not considered at the moment. It is assumed then that for $r_{\mathrm{i}} = 0$ and $s = 1$, the suspension occupies the whole of the container for all times and no interface between clear fluid and mixture forms. Equation (5.1) must be solved subject to the boundary condition (4.9), which implies

$$\phi^0 = 0, \tag{5.2}$$

at the vertical walls. This is easily done because the right-hand side of (5.1) is independent of space coordinates in the whole region S. Let

$$\psi(r,\,\theta) = \phi^0/2\omega^0, \tag{5.3}$$

so that from (5.1) and (5.2)

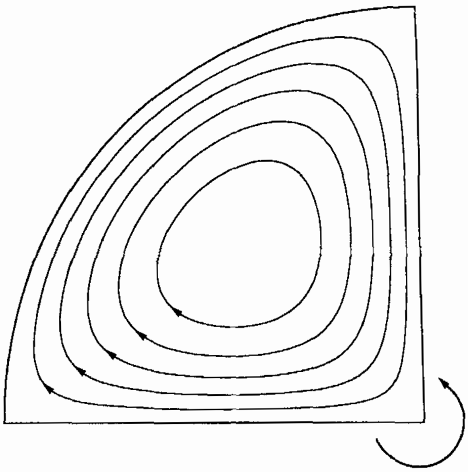
$$\nabla^2\psi = 1 \quad \text{in} \quad S, \qquad \psi = 0 \quad \text{on} \quad \partial S. \tag{5.4}$$

This can be solved by using a Green function (see Appendix A) for the cases of $\Theta = \frac{1}{2}\pi$ and $\Theta = \pi$; streamlines are shown in figure 4. With $\boldsymbol{q}^0$ determined, the velocity of the particles is obtained from (2.3*a*) and (3.4):

$$\boldsymbol{q}^0_{\mathrm{D}} = \boldsymbol{q}^0+\beta(1-\alpha^0)\,\frac{A^0}{1+(B^0)^2}\,r[\boldsymbol{e}_r+B^0\boldsymbol{e}_r\times\boldsymbol{k}]. \tag{5.5}$$

† It is interesting to note that in contrast to the 'mixture model' used here, the 'two-fluid' model (see Ishii 1975) has been applied by Ungarish (1988) to obtain similar results for non-zero Rossby numbers in a circular cylinder using numerical methods.

(a)



(b)

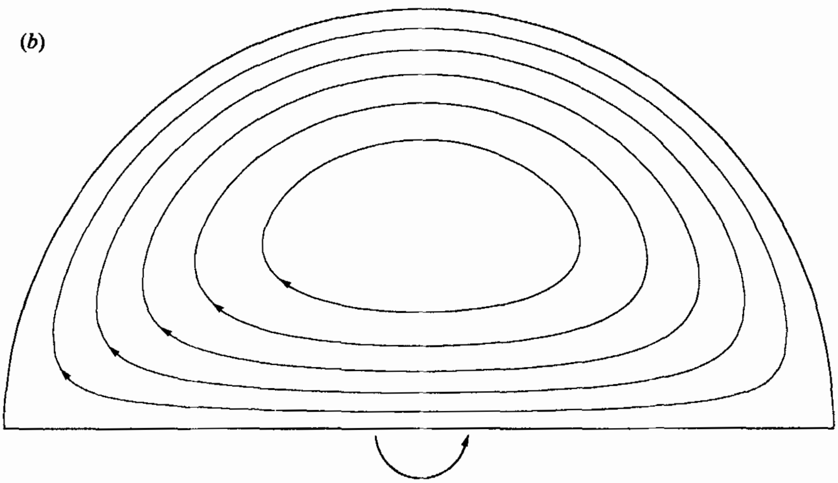


FIGURE 4. Streamlines for the mixture velocity given by the solution to (5.4) for (a) $\Theta = \frac{1}{2}\pi$, $\psi = -n \times 0.0092$, $n = 1, 2, \ldots, 5$; (b) $\Theta = \pi$, $\psi = -n \times 0.016$.

For small β then, the velocity of the particles deviates only slightly from the mixture velocity. It follows from (4.18), (4.20) and (4.1) that for large λ

$$q^0 \sim 1/\lambda = c\beta H, \tag{5.6}$$

which implies that the convection of particles decreases as λ increases, and this is clearly indicated in figure 5.

Consider next a container with an inner cylinder, in which case a core of clear fluid, C, is formed as the particles move away from the interior boundary (see figure 6). If in addition the azimuthal component of the relative velocity is accounted for, i.e. $\beta \neq 0$, particles also move away from the front wall at $\theta = \Theta$. The clear fluid produced there is assumed to be transported radially inwards in a quasi-steady boundary layer in which the mass flux of particles is negligible (Schaflinger *et al.* 1986). This implies the condition

$$q_D \cdot n = 0 \quad \text{at} \quad \partial S_D = \{r : \theta = \Theta,\ R(\Theta, t) \leqslant r \leqslant 1\} \tag{5.7}$$

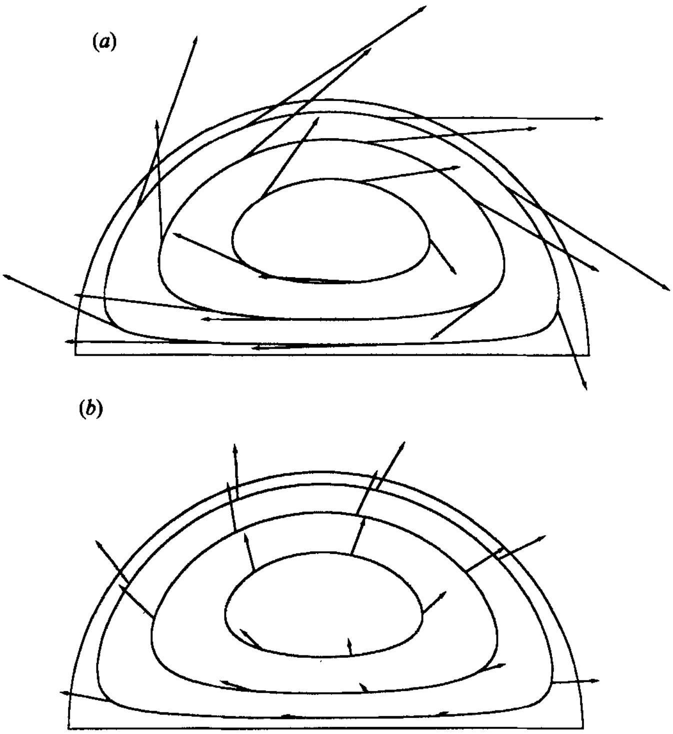


FIGURE 5. Particle velocities indicated with arrows on the streamlines of the mixture velocity: $t = 0.5$, $\alpha_1 = 0.1$, $\beta = 0.01$; (a) $\lambda = 0.1$, (b) $\lambda = 10.0$.

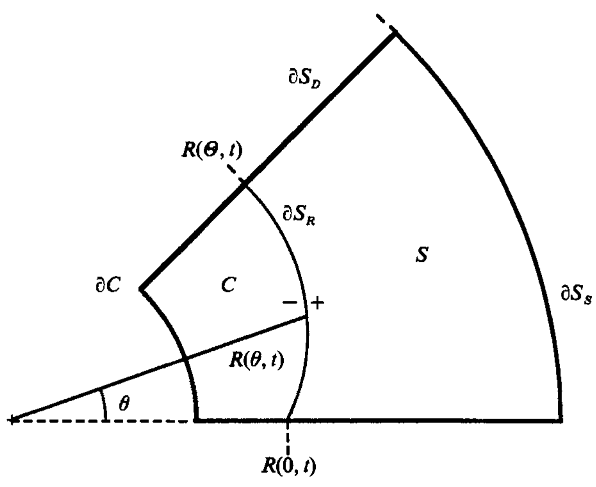


FIGURE 6. Geometry and notation.

(Schneider 1982), or by (2.3*a*),

$$q_\theta = -\beta \frac{1-\alpha}{1+\epsilon\alpha} \boldsymbol{q}_{\mathrm{R}} \cdot \boldsymbol{e}_\theta \quad \text{at} \quad \partial S_D. \tag{5.8}$$

The boundary conditions to zeroth order can be expressed in terms of P^0 and ϕ^0. At the clear-fluid boundary, the stream function in the clear fluid P^0 is constant and we take

$$P^0 = 0 \quad \text{at} \quad \partial C. \tag{5.9}$$

Since the pressure P is continuous to lowest order at $r = R(\theta, t)$, it follows from the definition (4.13) that in the suspension

$$\phi^0 = -\alpha^0(R^0(0,t))^2/2 \quad \text{at} \quad \partial S_S = \{\boldsymbol{r}: r = 1, \quad \boldsymbol{r}: \theta = 0, R(0,t) \leqslant r \leqslant 1\}, \tag{5.10}$$

and
$$\phi^0 = -\alpha^0(R^0(0,t))^2/2 - \frac{(1-r^2)(1-\alpha^0)(B^0)^2}{2[1+(B^0)^2]} \quad \text{at} \quad \partial S_D. \tag{5.11}$$

In addition to these boundary conditions the shock conditions at the position of the interface between clear fluid and suspension have to be considered. To zeroth order (2.14*a–c*), (2.15*a, b*) imply

$$\left.\begin{aligned} &[P^0]^+_- = 0, && (5.12a)\\ &[\boldsymbol{q}^0]^+_- = \boldsymbol{k} \times \boldsymbol{n}^0_R \beta\alpha^0(\boldsymbol{q}^0_{\mathrm{R}})^+ \cdot (\boldsymbol{k} \times \boldsymbol{n}^0_{\mathrm{R}}), && (5.12b)\\ &\beta\frac{\partial R^0}{\partial t} - (\boldsymbol{q}^0)^+ \cdot \boldsymbol{n}^0_R = \beta(1-\alpha^0)(\boldsymbol{q}^0_{\mathrm{R}})^+ \cdot \boldsymbol{n}^0_R && (5.12c) \end{aligned}\right\} \quad \text{at} \quad r = R^0(\theta, t).$$

The motion of the interface is governed by (5.12*c*) and is thus coupled to the flow field. Although the force balance is not purely hydrostatic, the shape of the interface (to zeroth order) is still a concentric circle. To establish this note that at any position θ

$$\int_0^H \int_{r_i}^1 j_\theta \, \mathrm{d}r \, \mathrm{d}z - 0, \tag{5.13}$$

so that, in particular,
$$\int_{r_i}^1 q^0_\theta \, \mathrm{d}r = 0. \tag{5.14}$$

Manipulation of (4.12) and (4.13), and (5.14) yields

$$(P^0)^- - P^0(r = r_i) + \phi^0(r = 1) - (\phi^0)^+ = 0, \tag{5.15}$$

and
$$(\phi^0)^+ = (P^0)^+ - \alpha^0(R^0(\theta,t))^2/2, \tag{5.16}$$

whereas the boundary conditions (5.9) and (5.10), and (5.15) imply that

$$[P^0]^+_- = \alpha^0[(R^0(\theta,t))^2 - (R^0(0,t))^2]/2. \tag{5.17}$$

But according to (5.12*a*) the pressure is continuous across the interface which means that

$$R^0(\theta, t) = R^0(0, t), \tag{5.18}$$

i.e. the shape of the interface is independent of θ to zeroth order. Since $\boldsymbol{n}^0_R = \boldsymbol{e}_r$, it follows from (5.12*c*) that

$$\frac{\partial R^0}{\partial t} = \frac{1}{\beta}(q^0_r)^+ + (1-\alpha^0)\frac{A^0R^0}{1+(B^0)^2}. \tag{5.19}$$

As the last term in (5.19) is a function of t only $(q^0_r)^+$, must also be a function only of t if (5.18) is to be fulfilled. In other words, the radial velocity is independent of θ at the position of the shock. This result makes it possible to solve for the flow in the two regions (clear fluid and suspension) separately, with the shock position appearing only as a new circular boundary. At the interface then

$$q^0_r(R^0(t), \theta, t) = f(t), \tag{5.20}$$

which is valid on both sides of the interface because the radial velocity component is continuous at the shock. The function $f(t)$ must be determined from global continuity requirements. In terms of ϕ^0, the preceding equation can be expressed as

$$\phi^0(R^0(t),\theta,t) = -\alpha^0(R^0(t))^2/2 - 2R^0(t)f(t)\,\theta. \tag{5.21}$$

Continuity requires that in the suspension at $\theta = \Theta$, (5.21) be identical to (5.11) at $r = R^0(t)$ and this implies

$$f(t) = \frac{1-(R^0)^2}{4R^0\Theta}\,\frac{(1-\alpha^0)(B^0)^2}{1+(B^0)^2}. \tag{5.22}$$

The pressure at the interface is given by

$$P^0(R^0(t),\theta,t) = -2R^0(t)f(t)\,\theta, \tag{5.23}$$

and because there is a pure fluid jet, P^0 is not continuous at $(R^0(t), \Theta)$ since (5.9) holds at ∂C. The clear fluid coming from the boundary layer at ∂S_D is assumed to be distributed into the clear-fluid region in form of a point source. The locus of the interface is obtained then from (5.19), (5.20) and (5.22):

$$\frac{\mathrm{d}(R^0)^2}{\mathrm{d}t} = \frac{(1-(R^0)^2)(B^0)^2}{2\beta\Theta[1+(B^0)^2]} + \frac{2(R^0)^2(1-\alpha^0)A^0}{1+(B^0)^2}, \tag{5.24}$$

which is equivalent to the result obtained by Schaflinger *et al.* (1986).

Let $\phi' = \phi + \alpha^0(R^0)^2/2$ so that:

$$\nabla^2\phi' = 2\omega^0 \quad \text{in} \quad S, \tag{5.25a}$$

$$\phi' = 0 \quad \text{on} \quad \partial S_S, \tag{5.25b}$$

$$\phi' = -\frac{(1-r^2)(1-\alpha^0)(B^0)^2}{2[1+(B^0)^2]} \quad \text{on} \quad \partial S_D, \tag{5.25c}$$

$$\phi' = -\frac{\theta}{\Theta}(1-(R^0)^2)\frac{(1-\alpha^0)(B^0)^2}{2[1+(B^0)^2]} \quad \text{on} \quad \partial S_R = \{r : r = R^0(t), \quad \theta : 0 \leqslant \theta \leqslant \Theta\}. \tag{5.25d}$$

The velocity is given by

$$\boldsymbol{q}^0 = \boldsymbol{k} \times \nabla\phi'/2 \tag{5.26}$$

and the time appears only as a parameter. It is convenient to represent the function ϕ' as a linear combination of ϕ_ω and ϕ_β,

$$\phi' = 2\omega^0(t)\,\phi_\omega + \frac{(1-\alpha^0)(B^0(t))^2}{2[1+(B^0(t))^2]}\,\phi_\beta, \tag{5.27}$$

where

$$\nabla^2\phi_\omega = 1 \quad \text{in} \quad S, \tag{5.28a}$$

$$\phi_\omega = 0 \quad \text{on} \quad \partial S, \tag{5.28b}$$

$$\nabla^2\phi_\beta = 0 \quad \text{in} \quad S, \tag{5.29a}$$

$$\phi_\beta = -(1-r^2) \quad \text{on} \quad \partial S_D, \tag{5.29b}$$

$$\phi_\beta = -\frac{\theta}{\Theta}(1-(R^0)^2) \quad \text{on} \quad \partial S_R. \tag{5.29c}$$

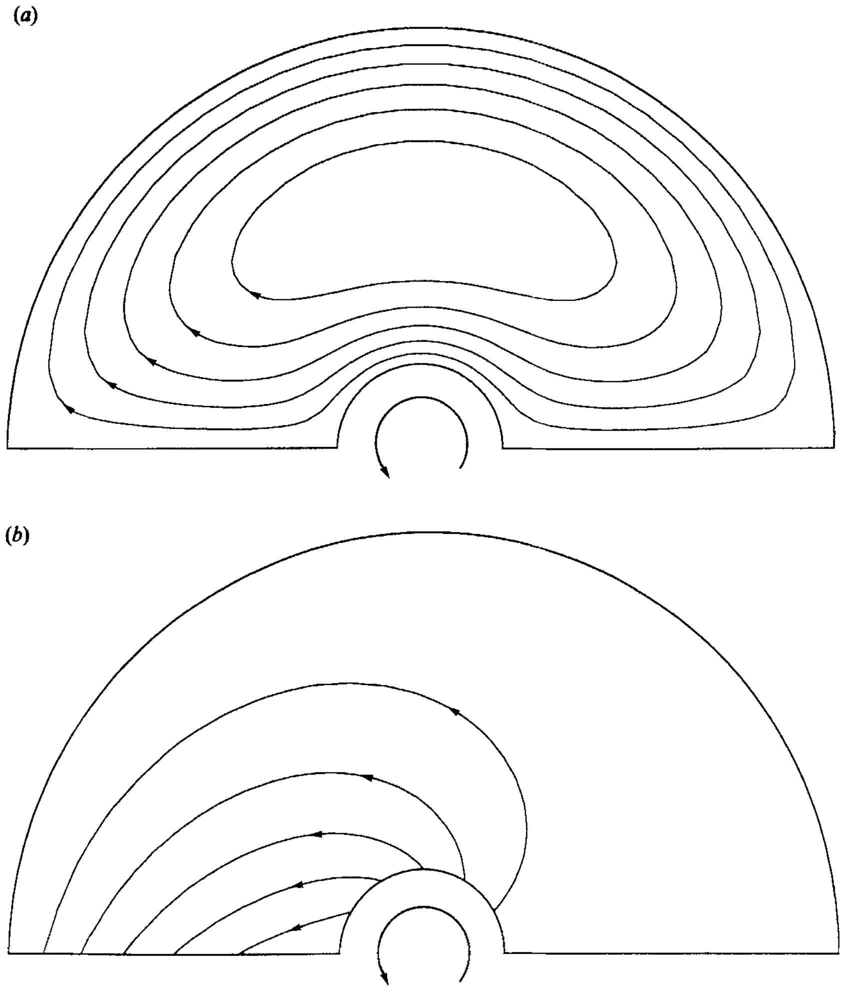


FIGURE 7. Streamlines in the mixture in the case of an inner interface with $R^0 = 0.2$: (a) Rotational part of the flow field, $\psi = -n \times 0.026$, $n = 1, 2, \ldots, 5$. (b) Irrotational part of the flow field, $\psi = -n \times 0.16$.

In this representation ϕ_ω is the part of the flow field that is set up by the generated vorticity, whereas ϕ_β is due to the suction of clear fluid at the front wall. Streamlines for the solution of (5.28) and (5.29) are shown separately in figures 7(a) and (b) for the case $\Theta = \pi$, $Ro = 0.2$. (Details of the method of solution can be found in Appendix B.) To determine the total flow field these solutions are superposed with relative magnitudes that are given by the time-dependent functions in (5.27). Since α_i and β are often rather small in applications, typically $\alpha_i = 0.1$, $\beta = 0.01$ or even less, it is interesting to estimate the relative importance of the two contributions to the flow field in practical cases. Since $B^0 \sim \beta$ from (2.8) and $\omega^0 \sim \alpha_i$ by (4.16) the ratio

$$A = \beta^2/\alpha_i, \tag{5.30}$$

measures the relative magnitude of the flow set up by ϕ_β which is very small in many applications. However, for $\lambda \gg 1$, $\omega^0 \sim \alpha_i/\lambda$ by (4.20) so that

$$A_\lambda = \beta^2\lambda/\alpha_i, \tag{5.31}$$

is a more accurate estimate indicating a more pronounced influence. (In this connection, it should be pointed out that the flow set up by ϕ_β does not possess any vorticity, and it is not damped by vortex stretching since the corresponding Ekman layers are non-divergent.)

No vorticity is generated in the clear fluid because there are no settling particles, and hence no generating mechanism. However, vorticity is convected along streamlines and since according to (5.19) the shock moves faster than the fluid itself, non-zero vorticity generated in the suspension is left behind the shock in the clear fluid as it moves outwards. In contrast to the situation in the suspension, the vorticity that appears in the clear fluid is not uniformly distributed because its magnitude crossing the shock is time dependent and in addition there is a decay due to spin-down processes. Therefore (5.1) and (4.16) cannot be solved separately as was the case in the suspension region, but a single equation for P^0 is obtained from (5.1), (4.12), (4.16) and (4.17):

$$\frac{\partial}{\partial t}\nabla^2 P^0 + \frac{1}{2\beta}(\boldsymbol{k}\times\nabla P^0)\cdot\nabla(\nabla^2 P^0) + 2\lambda[D(\alpha^0)]^{\frac{1}{2}}\nabla^2 P^0 = 0 \quad \text{in} \quad C. \tag{5.32}$$

The boundary conditions are

$$P^0 = 0 \quad \text{on} \quad \partial C, \tag{5.33a}$$

$$P^0 = -\frac{\theta}{\Theta}(1-(R^0)^2)\frac{(1-\alpha^0)(B^0)^2}{2[1+(B^0)^2]} \quad \text{on} \quad \partial S_R, \tag{5.33b}$$

and from the θ-component of (5.12b)

$$\frac{\partial P^0}{\partial r} = \frac{\partial \phi^0}{\partial r} + \alpha^0 R^0 \frac{(B^0)^2}{1+(B^0)^2} \quad \text{on} \quad \partial S_R, \tag{5.33c}$$

where ϕ^0 is the solution in the suspension region. The last boundary condition, which was not required earlier, is necessary here because the governing equation is of higher order than that in the suspension.

Some additional remarks should be made regarding the boundary condition (5.33c). This condition suggests a jump of the θ-component of the velocity across the shock, the magnitude of which is given by the last term in (5.33c). This term is a time-dependent function only and thus non-zero along the whole shock surface. At $\theta = 0$ though, the radial wall prevents motions in the θ-direction on both sides of the shock, and (5.33c) obviously cannot hold there. However, the magnitude of the velocity jump at the shock, $\sim \alpha_1^2 \Lambda$, is of the same order as the error made when neglecting the sediment, which coats the wall of the container at $\theta = 0$ as well as at $r = 1$. Therefore, the velocity jump at the shock must be dealt with in conjunction with a study of the sediment layer. Such a boundary-layer analysis would give the appropriate boundary conditions for the flow outside the sediment layer. The shock condition (5.33c) must then match these conditions rather than the condition at the wall. If, for example, the sediment is assumed to be transported away in a similar way as the clear fluid at $\theta = \Theta$, or the accumulation is of no consequence, the corresponding condition in the suspension at $\theta = 0$ would be $\boldsymbol{q}_C\cdot\boldsymbol{e}_\theta = 0$. This implies that in the suspension

$$\boldsymbol{q}^0\cdot\boldsymbol{e}_\theta = \frac{1}{2}\frac{\partial\phi^0}{\partial r} = -\alpha^0 r\frac{(B^0)^2}{2(1+(B^0)^2)} \quad \text{on} \quad \partial S_{S0} = \{\boldsymbol{r}: \theta = 0,\, R^0(t) \leqslant r \leqslant 1\}, \tag{5.34}$$

whereas in the clear fluid we still have $\boldsymbol{q}^0\cdot\boldsymbol{e}_\theta = 0$ at $\theta = 0$. Thus the velocity jump at the shock at $r = R^0(t)$ is in this case possible due to the suction of particles into the

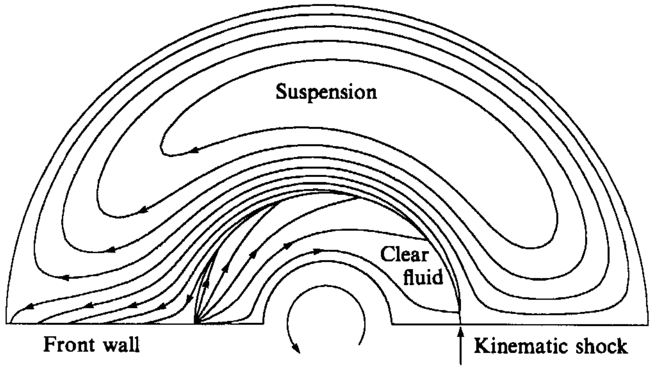


FIGURE 8. Streamlines in the mixture and in the clear fluid: $t = 0.7$, $\alpha_i = 0.1$, $\beta = 0.01$, $\lambda = 25$. Mixture: $\phi' = -0.11\times10^{-3}+n\times0.026\times10^{-3}$, $n = 0, 1, \ldots, 8$. Clear fluid: $P^0 = -0.11\times10^{-3}+n\times0.026\times10^{-3}$, $n = 0, 1, \ldots, 4$.

sediment layer in the suspension region. If, on the contrary, the sediment accumulation is significant, (5.34) does not hold and a more detailed study of the sediment layer is needed to formulate the proper boundary condition. This issue is not pursued further here though and (5.33*c*) is approximated by

$$\frac{\partial P^0}{\partial r} = \frac{\partial \phi^0}{\partial r} \quad \text{on} \quad \partial S_R. \tag{5.35}$$

The problem for the clear fluid is nonlinear and must be solved by numerical methods. However, an approximate analytical solution can be found for large values of the parameter λ for which the spin-up time is very small compared with the separation time. The vorticity that appears in the clear fluid is then damped out on a very short timescale. By neglecting all terms but the last in (5.32), the problem reduces to the same linear form as in the suspension, but now the vorticity is zero in the whole region. The vorticity which crosses the shock decays to zero almost immediately, that is within an inertial boundary layer of thickness $\delta \sim 1/\lambda$. A study of this boundary layer is necessary in order to satisfy the last condition (5.35), which cannot be satisfied by the outer solution presented here. (This will be reported on elsewhere.) We can now obtain a solution for the whole of the container in the case of large λ with the approximations already cited. Streamlines are shown in figure 8 at a time when the interface is approximately midway between the axis of rotation and the outer wall of the cylinder. The flow field has been calculated using the formula

$$\alpha^0 = \alpha_i \, e^{-2t}, \tag{5.36}$$

which is an approximation for dilute suspensions.

6. Conclusions

Batch separation of a two-phase mixture contained in a rotating sectioned cylinder of finite height has been considered. The two-phase flow equations have been solved under the assumption that the Ekman number and the relative density difference are small.

During separation, a bulk mass element in the mixture expands in the horizontal plane as the heavier phase is spread out by the centrifugal acceleration. To maintain its angular momentum, the mass element, which initially rotates with the container, acquires a relative rotation in the opposite direction. This is equivalent to a locally generated, negative relative vorticity in the mixture. For a non-stratified mixture at small Rossby number the expansion rate is spatially uniform so that only a time-dependent vorticity is generated. At the same time, the stretching of vortex lines due to the divergent Ekman layers at the endwalls counteracts the expansion of the mass elements, which has a damping effect on the generated vorticity. Diffusion of momentum caused by the sedimenting particles also reduces the vorticity that is generated in the mixture. In summary, this extends the result for the mixture vorticity in a circular cylinder, found by Ungarish (1986) to be valid for a cylinder with arbitrary cross-section.

The concentration remains approximately homogeneous for small Rossby numbers even in a non-axisymmetric cylinder. The first-order correction to the concentration for non-zero Rossby numbers results from the Coriolis force in the bulk which decreases the expansion rate of the particle cloud. The negative effect of the Coriolis force on the separation rate for increasing Rossby numbers thus persists even in a non-axisymmetric container.

The generated non-axisymmetric flow field in the mixture is approximately geostrophic and can be seen as a superposition of two parts. One part is due to the generated vorticity in the mixture and consists of a circulating motion opposite to the rotation of the container. The second part is caused by suction of clear fluid at the front wall. This fluid is redistributed into the clear-fluid region via a thin boundary layer. The importance of this part of the flow field is governed by the particle Taylor number and decreases the time for total separation according to the result found by Schaflinger *et al.* (1986).

The interface separating the regions of clear fluid and mixture is found to be approximately circular and is not influenced by the flow field to zeroth order in the Rossby number.

The perturbation technique used here has also been applied to separation in containers without closed geostrophic contours, in which the endwalls are slightly inclined. This investigation will be reported upon separately.

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Appendix A. Solution for the stream function in a circle sector

An analytical solution to (5.4) can be found using the Green function method. For the case $\Theta = \frac{1}{2}\pi$, it follows that

$$\begin{aligned}\psi(r,\theta) = \frac{r^2}{4}+\frac{1}{4\pi}\Bigg\{&\pi-2\,\mathrm{artan}\left[\frac{1+r^2}{1-r^2}\tan(\theta)\right]-2\,\mathrm{artan}\left[\frac{1+r^2}{1-r^2}\tan(\tfrac{1}{2}\pi-\theta)\right]\\
&-\tfrac{1}{2}r^2\sin(2\theta)\log\left[\left(\frac{1}{r^4}+1\right)^2-\frac{4}{r^4}\cos^2(2\theta)\right]+\frac{\sin(2\theta)}{2r^2}\log\left[(r^4+1)^2-4r^4\cos^2(2\theta)\right]\\
&-\frac{\cos(2\theta)}{r^2}\left(\mathrm{artan}\left[\frac{r^2+\cos(2\theta)}{\sin(2\theta)}\right]-2\,\mathrm{artan}\left[\frac{1}{\tan(2\theta)}\right]-\mathrm{artan}\left[\frac{r^2-\cos(2\theta)}{\sin(2\theta)}\right]\right)\\
&+r^2\cos(2\theta)\left(\mathrm{artan}\left[\frac{1/r^2+\cos(2\theta)}{\sin(2\theta)}\right]-2\,\mathrm{artan}\left[\frac{1}{\tan(2\theta)}\right]\right.\\
&\left.-\mathrm{artan}\left[\frac{1/r^2-\cos(2\theta)}{\sin(2\theta)}\right]\right)\Bigg\}. \qquad \text{(A 1)}\end{aligned}$$

For the case $\Theta = \pi$,

$$\psi(r,\theta) = \tfrac{1}{4}r^2 + \frac{1}{4\pi}\left\{\pi - 2\,\mathrm{artan}\left[\frac{1+r}{1-r}\tan(\tfrac{1}{2}\pi-\theta)\right] - 2\,\mathrm{artan}\left[\frac{1+r}{1-r}\tan(\tfrac{1}{2}\theta)\right]\right.$$
$$+2\sin(\theta)\left(\frac{1}{r}-r\right)+\tfrac{1}{2}\sin(2\theta)\left(\frac{1}{r^2}-r^2\right)\log\left[\frac{r^2+1-2r\cos(\theta)}{r^2+1+2r\cos(\theta)}\right]$$
$$-r^2\cos(2\theta)\left(\mathrm{artan}\left[\frac{1/r+\cos(\theta)}{\sin(\theta)}\right]+\mathrm{artan}\left[\frac{1/r-\cos(\theta)}{\sin(\theta)}\right]\right)$$
$$\left.+\frac{\cos(2\theta)}{r^2}\left(\mathrm{artan}\left[\frac{r+\cos(\theta)}{\sin(\theta)}\right]+\mathrm{artan}\left[\frac{r-\cos(\theta)}{\sin(\theta)}\right]\right)\right\}. \qquad \text{(A 2)}$$

Appendix B. The Green function for a semi-annulus

Equations (5.28) and (5.29) constitute a problem of the type

$$\nabla^2\phi = 0 \quad \text{in} \quad S; \quad \phi = f(r,\theta) \quad \text{on} \quad \partial S. \qquad \text{(B 1)}$$

A Green function to the domain

$$S = \{r: a \leqslant r \leqslant 1, \quad \theta: 0 \leqslant \theta \leqslant \pi\}$$

is found from an infinite set of mirror images of a point in S. In complex form the Green function can be written

$$\mathscr{G}(z,z_0) = \frac{1}{2\pi}\log\left\{\frac{B(z_0/z)-(z/z_0)B(z/z_0)}{B(1/z\bar{z}_0)-z\bar{z}_0B(z\bar{z}_0)}\right\} - \frac{1}{2\pi}\log\left\{\frac{B(\bar{z}_0/z)-(z/\bar{z}_0)B(z/\bar{z}_0)}{B(1/zz_0)-zz_0B(zz_0)}\right\}, \qquad \text{(B 2)}$$

where $B(w)$ is defined by

$$B(w) = \sum_{n=0}^{\infty}(-1)^n a^{n^2+n} w^n. \qquad \text{(B 3)}$$

Here, an overbar denotes the complex conjugate. The solution to (B 1) is then given by

$$\phi(\boldsymbol{r}_0) = \int_{\partial S} f(r,\theta)\,\mathrm{Re}\left\{\frac{\mathrm{d}\mathscr{G}}{\mathrm{d}z}\,n\right\}\mathrm{d}s, \qquad \text{(B 4)}$$

where n is the unit normal to ∂S in the complex plane. Solutions for other values of Θ than π can be found by conformal mapping.

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